Exercises for Quiz 1

Please have a look at the following problems. You may as well work on it as revision. I will present the solutions in the tutorial before the quiz.

Question 1. Suppose that $\lim a_n = 3$. Show that

$$\lim_{n \to \infty} \frac{a_n^2 + 1}{a_n - 2} = 10.$$

Question 2. Let (x_n) be a sequence of non-negative numbers. Suppose that

$$\lim_{n \to \infty} (-1)^n x_n$$

exists in $\mathbb{R}$. Show that (x_n) converges and find its limit.

Question 3. Let (x_n) be a bounded sequence of real numbers. Define

$$E = \{x \in \mathbb{R} : \text{there is a subseqence } (x_{n_k}) \text{ of } (x_n) \text{ that converges to } x.\}$$

Let $\alpha = \overline{\lim} \, a_n$. Show that $\alpha \in E$ and $\alpha = \sup E$.

Remark. The existence of α is guaranteed because E is non-empty by Bolzano-Weierstrass Theorem.

Question 4. Let a be a positive real number and $x_1 > \sqrt{a}$. Define the sequence (x_n) by

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

Prove that (x_n) is convergent and $\lim x_n = \sqrt{a}$.

Remark. This is the Newton's method of finding square roots.

Question 5. Prove the **Nested Interval Property** by using **Bolzano-Weierstrass Theorem**.

Remark. In the textbook, **Bolzano-Weierstrass Theorem** is proved using the **Nested Interval Property**.